

# Integral Presentations for the Universal $\mathcal{R}$ -matrix

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*to the memory of Denis Uglov*

## Abstract

We present an integral formula for the universal  $\mathcal{R}$ -matrix of quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$  with 'Drinfeld comultiplication'. We show that the properties of the universal  $\mathcal{R}$ -matrix follow from the factorization properties of the cycles in proper configuration spaces. For general  $\mathfrak{g}$  we conjecture that such cycles exist and unique. For  $U_q(\widehat{\mathfrak{sl}}_2)$  we describe precisely the cycles and present a new simple expression for the universal  $\mathcal{R}$ -matrix as a result of calculation of corresponding integrals.

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# 1 Introduction

The Yang-Baxter (YB) equation is one of cornerstones in the investigations of quantum integrable systems. The most important solutions to the YB equation were found by Yang, Baxter [B] and Zamolodchikovs [Z]. The mathematical background to the application of the YB equation was established in the theory of quantum groups by Drinfeld [D1] and Jimbo [J], based on the quantum inverse scattering method developed by Leningrad school [F].

The main object in this theory is a quantum group which is a Hopf algebra deformation of the universal enveloping algebra of contragredient Lie algebra. This Hopf algebra comes together with the universal  $R$ -matrix which acts in a tensor category of representations of quantum group and any such a specialization of the universal  $R$ -matrix provides a solution of the YB equation. The deformations of affine Lie algebras : Yangians, quantum affine algebras and their elliptic analogs are especially important: they produce  $R$ -matrices depending on a spectral parameter, in particular the main solutions of the YB equation listed above.

However,  $R$ -matrices and fundamental  $L$ -operators appear in quite different manner in physical and mathematical literature. In the works, written by physicists, we have their expressions written as ordered exponential integrals like a series of iterated integrals [BLZ], while in the mathematical literature all the known expressions can be reduced to products of certain  $q$ -exponential factors [KT]. This reflects the structure of quantum affine algebras: they deform the structure of contragredient Lie algebra for affine Lie algebras and do not touch actually their functional realization.

Drinfeld suggested another ('new') realization of quantum affine algebras and of Yangians [D], which, together with naturally attached coproduct structure, can be regarded as a Hopf algebra deformation of affine Lie algebras in a sense that it deforms their functional realization. This approach was successfully exploited for the study of finite-dimensional representations of quantum affine algebras and of Yangians (see [CP] and references therein) and for bosonization of their infinite-dimensional representations [FJ].

In [DK] a functional version of braid group technique was developed for quantum affine algebras and for the doubles of the Yangians. In particular, the currents corresponding to nonsimple roots of underlying Lie algebras were defined and a presentation of the universal  $R$ -matrix which uses multiple contour integrals was given.

In this note we develop further the ideas of [DK]. The first main result is a geometric reformulation of the properties of the universal  $R$ -matrix for quantum affine algebras equipped with so called Drinfeld comultiplication. Namely, we attach to any simple Lie algebra  $\mathfrak{g}$  a system of configuration spaces, which are complex linear spaces without shifted diagonals and coordinate hyperplanes. We claim that the defining properties of the universal  $R$ -matrix can be reduced to certain factorization properties for the homology classes of the cycles in configuration space. The cycles are actually relative: we are interested in their pairing with the forms satisfying certain zero residues conditions. In algebra, these vanishing conditions come from the Serre relations. Thus we have a purely homological problem which resembles the constructions of factorizable sheaves [BFS] and of factorizable  $D$ -modules [KS]. In general case we do not give here a construction of factorizable system of cycles, just formulate a conjecture of its existence.

Second result is the explicit calculation of the  $R$ -matrix for  $U_q(\widehat{\mathfrak{sl}}_2)$ . We suggest in this case the factorizable system of cycles in configuration space, generalizing the ideas of [DK] where it was done for  $|q| > 1$  and calculate the corresponding integrals. The calculations go as follows: we take some residues to derive the recurrence relations for  $n$ -th fold integral, which lead to a simple linear differential equation for the  $R$ -matrix. Its solution has a form of vertex operator over total integrals of residue currents. These currents appeared already in the study of integrable representations of  $U_q(\widehat{\mathfrak{sl}}_2)$  [DM]: the annihilator of level  $k$  integrable representations is described in their terms. As a consequence, the  $R$ -matrix in this case contains only finitely many currents which contribute into vertex operator, for level one it is just an exponent of total integral of the first one. An application of these results to quantum affine algebra  $U_q(\widehat{\mathfrak{sl}}_2)$  with traditional comultiplication is given in [DKP].

## 2 Quantized current algebras

By quantized current algebra we mean a completion of quantum affine algebra in Drinfeld "new" realization equipped with coproduct structure, introduced by Drinfeld in the deriving of this realization [D]. The completion

is done with respect to a minimal topology, in which the action in the graded modules with bounded from above degree is continuous.

## 2.1 "New" realization of quantum affine algebras

Let  $\mathfrak{g}$  be a simple Lie algebra. Following Drinfeld [D], we can describe quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$  as an algebra generated by the central element  $c$ , gradation operator  $d$  and by the modes of the currents  $e_{\pm\alpha_i}(z)$  and  $\psi_{\alpha_i}^{\pm}(z)$ ,

$$e_{\pm\alpha_i}(z) = \sum_{k \in \mathbb{Z}} e_{\pm\alpha_i, k} z^{-k}, \quad \psi_{\alpha_i}^{\pm}(z) = k_{\alpha_i}^{\pm 1} \exp \left( \pm (q - q^{-1}) \sum_{n > 0} a_{i, \pm n} z^{\mp n} \right), \quad (2.1)$$

where  $\alpha_i$ ,  $i = 1, \dots, r$  are positive simple roots of the algebra  $\mathfrak{g}$  of the rank  $r$ , and  $(\alpha_i, \alpha_j)$  is symmetrized Cartan matrix.

The generating functions of the quantum current algebra  $U_q(\widehat{\mathfrak{g}})$  satisfy the following defining relations:

$$q^d x(z) q^{-d} = x(qz), \quad \text{for } x = e_{\pm\alpha_i}, \psi_{\alpha_i}^{\pm}, \quad (2.2)$$

$$(z - q^{\pm(\alpha_i, \alpha_j)} w) e_{\pm\alpha_i}(z) e_{\pm\alpha_j}(w) = e_{\pm\alpha_j}(w) e_{\pm\alpha_i}(z) (q^{\pm(\alpha_i, \alpha_j)} z - w), \quad (2.3)$$

$$\frac{(q^{\pm c/2} z - q^{\pm(\alpha_i, \alpha_j)} w)}{(q^{\pm(\alpha_i, \alpha_j) \pm c/2} z - w)} \psi_{\alpha_i}^{\pm}(z) e_{\pm\alpha_j}(w) = e_{\pm\alpha_j}(w) \psi_{\alpha_i}^{\pm}(z), \quad (2.4)$$

$$\psi_{\alpha_i}^{\pm}(z) e_{\pm\alpha_j}(w) = \frac{(q^{\pm(\alpha_i, \alpha_j) \mp c/2} z - w)}{(q^{\mp c/2} z - q^{\pm(\alpha_i, \alpha_j)} w)} e_{\pm\alpha_j}(w) \psi_{\alpha_i}^{\pm}(z), \quad (2.5)$$

$$\frac{(z - q^{(\alpha_i, \alpha_j) - c} w)(z - q^{-(\alpha_i, \alpha_j) + c} w)}{(q^{(\alpha_i, \alpha_j) + c} z - w)(q^{-(\alpha_i, \alpha_j) - c} z - w)} \psi_{\alpha_i}^{\pm}(z) \psi_{\alpha_j}^{\mp}(w) = \psi_{\alpha_j}^{\mp}(w) \psi_{\alpha_i}^{\pm}(z), \quad (2.6)$$

$$\psi_{\alpha_i}^{\pm}(z) \psi_{\alpha_j}^{\pm}(w) = \psi_{\alpha_j}^{\pm}(w) \psi_{\alpha_i}^{\pm}(z), \quad (2.7)$$

$$[e_{\alpha_i}(z), e_{-\alpha_j}(w)] = \frac{\delta_{\alpha_i, \alpha_j}}{q - q^{-1}} \left( \delta(z/q^c w) \psi_{\alpha_i}^+(z q^{-c/2}) - \delta(z q^c / w) \psi_{\alpha_i}^-(w q^{-c/2}) \right) \quad (2.8)$$

and the Serre relations:

$$\sum_{r=0}^{n_{ij}} (-1)^r \begin{bmatrix} n_{ij} \\ r \end{bmatrix}_{q_{\alpha_i}} \text{Sym}_{z_1, \dots, z_{n_{ij}}} e_{\pm\alpha_i}(z_1) \cdots e_{\pm\alpha_i}(z_r) e_{\pm\alpha_j}(w) e_{\pm\alpha_i}(z_{r+1}) \cdots e_{\pm\alpha_i}(z_{n_{ij}}) = 0 \quad (2.9)$$

for any simple roots  $\alpha_i \neq \alpha_j$ . Here  $\delta(z) = \sum_{k \in \mathbb{Z}} z^k$ ,  $n_{ij} = 1 - a_{ij}$ , where  $a_{ij}$  is  $(i, j)$  entry of Cartan matrix of  $\mathfrak{g}$ ,  $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!}$ ,  $[n]_q! = [1]_q [2]_q \cdots [n]_q$ ,  $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ ,  $q_{\alpha} = q^{\frac{(\alpha, \alpha)}{2}}$ .

The formulas (2.4)–(2.6) can be rewritten also using the generators  $a_{i, n}$  introduced by (2.1):

$$[a_{i, n}, e_{\pm\alpha_j}(z)] = \pm \frac{[(\alpha_i, \alpha_j) n]_q}{n} q^{\mp c|n|/2} z^n e_{\pm\alpha_j}(z), \quad (2.10)$$

$$[a_{i, n}, a_{j, m}] = \delta_{n+m, 0} \frac{[(\alpha_i, \alpha_j) n]_q [cn]_q}{n}. \quad (2.11)$$

We assign to this algebra two Drinfeld type Hopf structures:

$$\Delta^{(1)} e_{\alpha_i}(z) = e_{\alpha_i}(z) \otimes 1 + \psi_{\alpha_i}^-(z q^{c_1/2}) \otimes e_{\alpha_i}(z q^{c_1}), \quad (2.12a)$$

$$\Delta^{(1)} e_{-\alpha_i}(z) = 1 \otimes e_{-\alpha_i}(z) + e_{-\alpha_i}(z q^{c_2}) \otimes \psi_{\alpha_i}^+(z q^{c_2/2}), \quad (2.12b)$$

$$\Delta^{(2)} e_{\alpha_i}(z) = e_{\alpha_i}(z) \otimes 1 + \psi_{\alpha_i}^+(z q^{-c_1/2}) \otimes e_{\alpha_i}(z q^{-c_1}), \quad (2.13a)$$

$$\Delta^{(2)} e_{-\alpha_i}(z) = 1 \otimes e_{-\alpha_i}(z) + e_{-\alpha_i}(zq^{-c_2}) \otimes \psi_{\alpha_i}^-(zq^{-c_2/2}), \quad (2.13b)$$

$$\Delta^{(1)} \psi_{\alpha_i}^\pm(z) = \Delta^{(2)} \psi_{\alpha_i}^\pm(z) = \psi_{\alpha_i}^\pm(zq^{\pm c_2/2}) \otimes \psi_{\alpha_i}^\pm(zq^{\mp c_1/2}), \quad (2.14)$$

with corresponding expressions for counit and antipode maps. One can see that the comultiplication operators map an algebra  $U_q(\widehat{\mathfrak{g}})$  to a completed tensor product  $U_q(\widehat{\mathfrak{g}}) \hat{\otimes} U_q(\widehat{\mathfrak{g}})$ . One possibility to describe it is to use in the right hand side the Taylor extension of  $U_q(\widehat{\mathfrak{g}}) \otimes U_q(\widehat{\mathfrak{g}})$ , defined in [KT1]. Below we define instead a completion in a weaker topology, adapted for the action on highest degree representations.

## 2.2 The completion

Let  $U = U_q(\widehat{\mathfrak{g}})$ . For any  $x \in U$ , we denote  $|x| < k$  if  $x$  can be presented as a noncommutative polynomial of degree less then  $k$  over the generators of the algebra, which are listed above.

The grading element  $d$  defines a gradation on  $U$ :  $\deg x = r$ , if  $[d, x] = rx$ . Let  $U_r$  be a linear subspace of  $U$ , generated by the elements of degree  $r$  and  $U^k$  be a linear subspace of  $U$  generated by all  $x \in U$ , such that  $|x| + |\deg x| < k$ . It is clear, that  $\mathbb{C}1 = U^1 \subset U^2 \subset U^3 \subset \dots$  and  $\cup U_k = U$ .

Define  $U_r^k$  as

$$U_r^k = U^k \cap \sum_{s \geq r} U \cdot U_s.$$

We denote by  $U^<$  a topological vector space  $U$  with  $U_r^k$  being basic open neighborhoods of zero, and by  $\overline{U}^<$  the corresponding completion.

One can note, that this topology can be defined in equivalent way by the system of open sets

$${}_r U^k = U^k \cap \sum_{s \leq r} U_s \cdot U.$$

Let  $W = U \otimes U$ . It is bigraded algebra, where  $\deg_1$  is defined by the action of  $d \otimes 1$ , and  $\deg_2$  is defined by the action of  $1 \otimes d$ . We consider  $x_k \otimes 1$  and  $1 \otimes x_k$  as generators of  $W$ , where  $x_k$  are generators of  $U$ . In this setting we define  $W^k$  as linear span of all  $x \in W$  such that  $|\deg_1 x| + |\deg_2 x| + |x| < k$  and

$$W_r^k = W^k \cap (U_r^k \otimes U + U \otimes U_r^k). \quad (2.15)$$

We denote by  $W^<$  the corresponding topological vector space and by  $\overline{W}^<$  its completion. Let  $\Delta^{(1)}$  and  $\Delta^{(2)}$  (2.12) and (2.13) be two types of Drinfeld comultiplications in  $U_q(\widehat{\mathfrak{g}})$ .

**Proposition 2.1** (i) *The multiplication*

$$m : W^< \longrightarrow U^<$$

*is continuous map;*

(ii) *The comultiplications*

$$\Delta^{(1)}, \Delta^{(2)} : U^< \longrightarrow \overline{W}^<$$

*are continuous maps;*

(iii) *The completed algebra  $\overline{U}^<$  acts on the highest degree modules over  $U$ ;*

(iv) *The completed algebra  $\overline{W}^<$  acts on the tensor products of highest degree modules over  $U$ .*

Here by the highest degree representation we mean graded (over operator  $d$ ) representation  $V$ , such that the graded components of  $V$  vanish for big enough degree:

$$V = \sum_{k < N} V_k.$$

For instance, all highest weight representations of  $U_q(\widehat{\mathfrak{g}})$  are of highest degree.

Actually, ‘ $<$ ’ is the smallest topology, in which the action on any highest degree representation is continuous. We will use this property further checking the assertions on highest degree modules. Moreover, as we see from the proposition, the completion  $\overline{U}^<$  is well defined topological Hopf algebra in the category of highest degree representations.

In quite analogous manner, we can define the topological Hopf algebra  $U^>$  by means of open sets

$$\tilde{U}_r^k = U^k \cap \sum_{s \leq r} U \cdot U_s. \quad (2.16)$$

Then its completion  $\overline{U}^>$  is well defined topological Hopf algebra in the category of lowest degree representations. Denote further the completed algebra  $\overline{U}^<$  equipped with comultiplication  $\Delta^{(1)}$  as  $U_q^{(D)}(\widehat{\mathfrak{g}})$ .

The described completion  $U_q^{(D)}(\widehat{\mathfrak{g}})$  of the algebra  $U_q(\widehat{\mathfrak{g}})$  changes its algebraical properties. In particular, it cannot be any more treated as a double of its Hopf subalgebra, generated by  $e_{-\alpha_i, n}$ ,  $n \in \mathbb{Z}$  and  $a_{i, k}$ ,  $k \geq 0$ , since in the dual subalgebra the topology should be reversed. In this point we change the ideology of [DK]. However, unlike the known examples, the Hopf algebra  $U_q^{(D)}(\widehat{\mathfrak{g}})$  possesses the universal  $R$ -matrix, which acts in tensor category of its representations.

### 3 Universal $\mathcal{R}$ -matrices

The theory of Cartan-Weyl basis for quantum affine algebras, developed in [TK], [KT1], [KT], [Be] gives possibility to write down the universal  $R$ -matrix for quantum affine algebras equipped with Drinfeld’s comultiplication, in a form of infinite product over Cartan-Weyl generators. It was done in [KT]. This formula is quite effective for  $U_q(\widehat{\mathfrak{sl}}_2)$  but does not look nice for higher rank. We remind it below. Next, we analyze the integral form of the  $R$ -matrix for the completed algebra  $U_q^{(D)}(\widehat{\mathfrak{sl}}_2)$ , presented in [DK] and observe the geometric properties of the cycles in certain configuration space, responsible for the properties of the universal  $R$ -matrix. As a corollary, we suggest the generalization of the formula in [DK] for other  $U_q^{(D)}(\widehat{\mathfrak{g}})$ . The next section is devoted to explicit calculations in  $U_q^{(D)}(\widehat{\mathfrak{sl}}_2)$ .

#### 3.1 The multiplicative formula [KT]

In the study of twistings of quantum affine algebra by means of the elements of the braid group the multiplicative formula for the universal  $R$ -matrix for Drinfeld comultiplications was suggested in [KT]. It was written by means of Cartan-Weyl generators and can be described as follows.

Let  $e_{\pm\alpha_i}, k_{\alpha_i}^{\pm 1}$ ,  $i = 0, 1, \dots, r$  be Chevalley generators of  $U_q(\widehat{\mathfrak{g}})$ , where  $r = \text{rank } \mathfrak{g}$ . For their connection to current generators  $e_{\pm\alpha_i, n}$ ,  $a_{i, n}$  see [D]. Let  $\underline{\Delta}$  be a system of positive roots of affine Lie algebra  $\widehat{\mathfrak{g}}$ . Let  $\alpha_0$  be an affine root and  $\delta$  be a minimal imaginary root. By a normal ordering of the root system  $\underline{\Delta}$  we mean a total linear order  $<$  of all the roots satisfying the condition: for any two roots  $\alpha$ ,  $\beta$ , such that at least one of them is real and  $\alpha + \beta$  is a root only two possibility of mutual position of  $\alpha$ ,  $\beta$ ,  $\alpha + \beta$  can occur:  $\alpha < \alpha + \beta < \beta$  or  $\beta < \alpha + \beta < \alpha$ . Let  $<$  be arbitrary normal ordering of the root system  $\underline{\Delta}$ , satisfying additional restriction  $\alpha_k < \delta < \alpha_0$  for any  $k = 1, \dots, r$ . The Cartan-Weyl generators are constructed as successive  $q$ -commutators by induction on the normal order. An induction step looks as follows. We put

$$e_{\alpha+\beta} = [e_{\alpha}, e_{\beta}]_q, \quad e_{-\alpha-\beta} = [e_{-\beta}, e_{-\alpha}]_{q^{-1}} \quad (3.1)$$

if  $e_{\alpha}$ ,  $e_{-\alpha}$ ,  $e_{\beta}$ ,  $e_{-\beta}$  are already defined,  $\alpha < \alpha + \beta < \beta$  and  $[\alpha, \beta]$  is a minimal segment, including  $\alpha + \beta$ , that is if there are other roots  $\alpha'$ ,  $\beta'$ , such that  $\alpha' + \beta' = \alpha + \beta$  and  $\alpha' < \alpha + \beta < \beta'$  then either  $\alpha' < \alpha$  or  $\beta < \beta'$ . The  $q$ -commutators  $[e_{\alpha}, e_{\beta}]_q$  and  $[e_{-\beta}, e_{-\alpha}]_{q^{-1}}$  mean

$$[e_{\alpha}, e_{\beta}]_q = e_{\alpha}e_{\beta} - q^{(\alpha, \beta)}e_{\beta}e_{\alpha}, \quad [e_{-\beta}, e_{-\alpha}]_{q^{-1}} = e_{-\beta}e_{-\alpha} - q^{-(\alpha, \beta)}e_{-\alpha}e_{-\beta}.$$

More precisely, we start from the simple roots and use the induction procedure (3.1) for the construction of

new *real* roots until we get the root vectors  $e_{\pm(\delta-\alpha_i)}$ ,  $i = 1, \dots, r$ . Then we put

$$\begin{aligned} e_{\delta}^{(i)} &= [e_{\alpha_i}, e_{\delta-\alpha_i}]_q, & e_{-\delta}^{(i)} &= [e_{\delta-\alpha_i}, e_{\alpha_i}]_{q^{-1}}, \\ e_{n\delta+\alpha_i} &= (-1)^n ([(\alpha_i, \alpha_i)]_q)^{-n} (\text{ad } e_{\delta}^{(i)})^n e_{\alpha_i}, & e_{-n\delta-\alpha_i} &= ([(\alpha_i, \alpha_i)]_q)^{-n} (\text{ad } e_{-\delta}^{(i)})^n e_{-\alpha_i}, \\ e_{(n+1)\delta-\alpha_i} &= ([(\alpha_i, \alpha_i)]_q)^{-n} (\text{ad } e_{\delta}^{(i)})^n e_{\delta-\alpha_i}, & e_{-(n+1)\delta+\alpha_i} &= (-1)^n ([(\alpha_i, \alpha_i)]_q)^{-n} (\text{ad } e_{-\delta}^{(i)})^n e_{-\delta+\alpha_i}, \\ e'_{(n+1)\delta}^{(i)} &= [e_{n\delta+\alpha_i}, e_{\delta-\alpha_i}]_q, & e'_{-(n+1)\delta}^{(i)} &= [e_{-\delta+\alpha_i}, e_{-n\delta-\alpha_i}]_{q^{-1}} \end{aligned}$$

(for  $n > 0$ ), where  $(\text{ad } x)y = [x, y]$  is a usual commutator.

The imaginary root vectors  $e_{\pm n\delta}^{(i)}$ , which coincide with  $a_{i,\pm n}$  up to central factors, are related to  $e'_{n\delta}^{(i)}$  via Schur polynomials, namely

$$E_{\pm i}(z) = \ln(1 + E'_{\pm i}(z)),$$

where

$$E_{\pm i}(z) = \pm(q - q^{-1}) \sum_{m \geq 1} e_{\pm m\delta}^{(i)} z^m, \quad E'_{\pm i}(z) = \pm(q - q^{-1}) \sum_{m \geq 1} e'_{\pm m\delta}^{(i)} z^m.$$

The rest of the real root vectors we construct in accordance with the induction procedure (3.1) using the root vectors  $e_{n\delta+\alpha_i}$ ,  $e_{(n+1)\delta-\alpha_i}$ ,  $e_{(n+1)\delta}^{(i)}$ , ( $i = 1, 2, \dots, r$ ;  $n \in \mathbb{Z}_+$ ). In this procedure the dependence on the choice of the last vector, that is on the choice of an index  $i$  is only in the normalization of the root vectors.

For any real root  $\gamma$  denote by  $R_{\gamma}$  the formal series

$$R_{\gamma} = \exp_{q(\gamma, \gamma)} (C_{\gamma}^{-1} e_{\gamma} \otimes e_{-\gamma})$$

where  $C_{\gamma}$  is the normalization constant given from the relation  $[e_{\gamma}, e_{-\gamma}] = C_{\gamma} (k_{\gamma} - k_{\gamma}^{-1})$  and

$$\exp_p(x) = 1 + \sum_{n > 0} \frac{x^n}{(n)_p!}, \quad (n)_p = \frac{p^n - 1}{p - 1}.$$

Put also

$$\mathcal{K} = q^{-t} q^{\frac{-c \otimes d - d \otimes c}{2}} \prod_{n > 0} \exp \left( -n(q - q^{-1}) \sum_{i,j=1}^r d_{i,j}^{(n)} a_{i,n} \otimes a_{j,-n} \right) q^{\frac{-c \otimes d - d \otimes c}{2}}, \quad (3.2)$$

where  $t = \sum h_i \otimes h^i$  is an invariant element in tensor square of Cartan subalgebra  $\mathfrak{h} \otimes \mathfrak{h}$  of  $\mathfrak{g}$ ;  $d_{i,j}^{(n)}$  is an inverse matrix to symmetrized Cartan matrix of  $\mathfrak{g}$

$$b_{i,j}^{(n)} = [n(\alpha_i, \alpha_j)]_q, \quad i, j = 1, \dots, r.$$

The universal  $R$ -matrix  $\mathcal{R}^{CW}$  for quantum affine algebra  $U_q(\widehat{\mathfrak{g}})$  with Drinfeld comultiplication  $\Delta^{(1)}$  was identified in [KT] with formal ordered infinite product

$$\mathcal{R}^{CW} = \mathcal{K} \mathcal{R}_-^{CW} \mathcal{R}_+^{CW}, \quad (3.3)$$

where

$$\mathcal{R}_-^{CW} = \overrightarrow{\prod}_{\gamma \in \underline{\Delta}^{re}, \gamma > \delta} R_{\gamma}, \quad \mathcal{R}_+^{CW} = \left( \overrightarrow{\prod}_{\gamma \in \underline{\Delta}^{re}, \gamma > \delta} R_{\gamma} \right)^{21}.$$

For instance, in the case of  $U_q(\widehat{\mathfrak{sl}}_2)$  the general expression (3.3) looks as

$$\mathcal{R}^{CW} = q^{-\frac{h \otimes h}{2}} q^{-c \otimes d - d \otimes c} \exp \left( (q^{-1} - q) \sum_{k > 0} \frac{n}{[2n]_q} q^{-\frac{cn}{2}} a_n \otimes q^{\frac{cn}{2}} a_{-n} \right) \cdot \overrightarrow{\prod}_{n \in \mathbb{Z}} \exp_{q^2} \left( (q^{-1} - q) f_{-n} \otimes e_n \right) \quad (3.4)$$

where we drop for simplicity the index of simple root and put  $e_n \equiv e_{\alpha, n}$ ,  $f_n \equiv e_{-\alpha, n}$ .

**Proposition 3.1** *The tensor  $\mathcal{R}^{CW}$  from (3.3) defines correctly determined operator in tensor product  $W \otimes V$  of  $U_q(\widehat{\mathfrak{g}})$ -modules if either  $W$  is a lowest weight representation or  $V$  is a highest weight representation.*

Let us consider a case when  $V$  is highest weight representation. Take two weight vectors  $v \in V$  and  $\xi \in V^*$ . Consider the matrix coefficient  $\langle \xi, \overline{\mathcal{R}}^{CW} v \rangle$ . We claim that only finitely many terms of formal series (3.3) contribute into this matrix coefficient. Indeed, let  $\rho \in \widehat{\mathfrak{h}}^*$ , where  $\widehat{\mathfrak{h}}$  is Cartan subalgebra of  $\widehat{\mathfrak{g}}$ , satisfy the conditions  $(\alpha_i, \rho) = 1$ ,  $i = 0, 1, \dots, r$ . Then, by definition of highest weight representation, there exists some big positive  $N$  such that  $xv = 0$  for any  $x \in U_q(\widehat{\mathfrak{g}})$ , such that  $(\lambda(x), \rho) > N$ , where  $\lambda(x)$  is a weight of  $x$ . All the entries from  $\mathcal{R}_+^{CW}$  have positive weight with respect to right tensor factor and there are only finitely many terms that satisfy to condition  $(\lambda(x), \rho) < N$ . Let us fix one such term  $x$ . We are restricted now to the summands of the type  $yx$ , where  $y \in \mathcal{KR}_-^{CW}$ , so for the  $y$  we have opposite restriction  $(\lambda(y), \rho) > -M$  for some positive  $M$  because  $\xi \in V^*$ . But the weights of the terms from  $\mathcal{KR}_-^{CW}$  have negative weights with respect to the right tensor factor, so by the same arguments we have only finitely many choices for  $y$ .

This shows that any matrix coefficient  $\langle \eta \otimes \xi, Rv \otimes v \rangle$ ,  $v \in V, \xi \in V^*, w \in W, \eta \in W^*$  is well define finite sum if  $V$  is of highest weight. Analogous arguments are valid when  $W$  is of lowest weight.

Proposition 3.1 allows to check the identities for  $\mathcal{R}^{CW}$  by their application onto highest weight or lowest weight representation. For instance, the equalities

$$\Delta^{op}(a) = \mathcal{R}^{CW} \Delta(a) (\mathcal{R}^{CW})^{-1},$$

$$(\Delta \otimes \text{id}) \mathcal{R}^{CW} = (\mathcal{R}^{CW})_{13} (\mathcal{R}^{CW})_{23}, \quad (\text{id} \otimes \Delta) \mathcal{R}^{CW} = (\mathcal{R}^{CW})_{13} (\mathcal{R}^{CW})_{12}.$$

are correct at least when we apply them to tensor products of highest (or lowest) weight representations.

### 3.2 Analytical properties of matrix coefficients [E]

Our considerations are strongly based on the analytical properties of the matrix coefficients of the products of the generating functions for  $U_q(\widehat{\mathfrak{g}})$  [E]. In the most general form they can be formulated as follows. Let  $a_k(z)$  stands for any of the generating functions of the type  $e_{\pm \alpha_i}(z)$ ,  $\psi_{\alpha_i}^{\pm}(z)$ ,  $V$  be a highest degree representation in a sense of the previous section,  $v \in V$  and  $\xi \in V^*$  be two homogeneous vectors. Consider the matrix coefficient

$$\langle \xi, a_{k_1}(z_1) \cdots a_{k_m}(z_m) v \rangle$$

as a formal power series over  $z_1, \dots, z_m$ .

Then, first, this formal power series belongs to a space

$$\mathbb{C}[z_1, z_1^{-1}, \dots, z_m, z_m^{-1}] \left[ \left[ \frac{z_2}{z_1}, \frac{z_3}{z_2}, \dots, \frac{z_m}{z_{m-1}} \right] \right], \quad (3.5)$$

that is, can be presented as Taylor series over the variables  $z_2/z_1, \dots, z_{m-1}/z_m$  with coefficients being polynomials over  $z_1, z_1^{-1}, \dots, z_m, z_m^{-1}$ .

Second, this formal power series converges in the region  $|z_1| \gg |z_2| \gg \dots \gg |z_m|$  to a rational function which poles in  $(\mathbb{C}^*)^m$  are dictated by the quadratic relations (2.3)–(2.8).

The proof of the first statement is based on the observation that under the conditions on the vectors  $v$  and  $\xi$  for some  $N$  big enough the coefficient at  $z_1^{-n_1} z_2^{-n_2} \cdots z_m^{-n_m}$  vanishes if  $n_m > N$ , or  $n_{m-1} + n_m > N$ , or  $n_{m-2} + n_{m-1} + n_m > N$  and so on due to the definition of highest degree representation.

Once (3.5) is established, we can repeat the arguments of [E] in order to show that the formal power series converges to a meromorphic function.

Consider, for instance, the matrix coefficient  $\langle \xi, e_{\alpha_{i_1}}(z_1) \cdots e_{\alpha_{i_m}}(z_m) v \rangle$ . We know from commutation relations that

$$\prod_{k < l} (z_k - q^{(\alpha_{i_k}, \alpha_{i_l})} z_l) e_{\alpha_{i_1}}(z_1) \cdots e_{\alpha_{i_m}}(z_m) = \prod_{k < l} (q^{(\alpha_{i_k}, \alpha_{i_l})} z_k - z_l) e_{\alpha_{i_m}}(z_m) \cdots e_{\alpha_{i_1}}(z_1), \quad (3.6)$$

so the matrix coefficient

$$\langle \xi, \prod_{k < l} (z_k - q^{(\alpha_{i_k}, \alpha_{i_l})} z_l) e_{\alpha_{i_1}}(z_1) \cdots e_{\alpha_{i_m}}(z_m) v \rangle$$

belongs to  $\mathbb{C}[z_1, z_1^{-1}, \dots, z_m, z_m^{-1}][[\frac{z_2}{z_1}, \frac{z_3}{z_2}, \dots, \frac{z_m}{z_{m-1}}]]$ ; but due to the form of the r.h.s. of (3.6), it belongs also to  $\mathbb{C}[z_1, z_1^{-1}, \dots, z_m, z_m^{-1}][[\frac{z_1}{z_2}, \frac{z_2}{z_3}, \dots, \frac{z_{m-1}}{z_m}]]$ , so lies in their intersection,  $\mathbb{C}[z_1, z_1^{-1}, \dots, z_m, z_m^{-1}]$ . It means that the series  $\langle \xi, e_{\alpha_{i_1}}(z_1) \cdots e_{\alpha_{i_m}}(z_m) v \rangle$  converges in the region  $|z_k| > |q^{(\alpha_{i_k}, \alpha_{i_l})} z_l|$  to a meromorphic in  $(\mathbb{C}^*)^m$  function with simple poles at  $z_k = q^{(\alpha_{i_k}, \alpha_{i_l})} z_l$ .

These properties of matrix coefficients allow to treat in the highest weight representation the generating functions  $a_{k_1}(z_1) \cdots a_{k_m}(z_m)$  as operator valued functions, analytical in the region  $|z_1| \gg |z_2| \gg \dots \gg |z_m|$ , where it coincides with the products  $(a_{k_1}(z_1) \cdots a_{k_l}(z_l)) \cdot (a_{k_{l+1}}(z_{l+1}) \cdots a_{k_m}(z_m))$  for any  $l: 1 \leq l \leq m$ . This is because the multiplication of Taylor series is well defined. Moreover, the relations (2.3)–(2.8) describe the analytical continuations of these functions to other regions. For instance, the analytical continuation of the function  $e_{\alpha_i}(z) e_{\alpha_j}(w)$  from the region  $|z| > |q^{(\alpha_i, \alpha_j)} w|$  to the region  $|z| < |q^{(\alpha_i, \alpha_j)} w|$  is defined, due to (2.3), by formal power series in the completed algebra  $U_q^{(D)}(\hat{\mathfrak{g}})$

$$\frac{q^{-(\alpha_i, \alpha_j)} - \frac{z}{w}}{1 - q^{-(\alpha_i, \alpha_j)} \frac{z}{w}} e_{\alpha_j}(w) e_{\alpha_i}(z).$$

### 3.3 The configuration spaces

We are interested now in the operator valued analytical functions, given as products (in the above sense) of the currents

$$t_{\alpha_i}(z) = (q^{-1} - q) e_{-\alpha_i}(z) \otimes e_{\alpha_i}(z).$$

We have, due to the relations (2.3):

The function  $t_{\alpha_{i_1}}(z_1) \cdots t_{\alpha_{i_n}}(z_n)$  is analytical in the region  $|z_k| > \max(|q^{\pm(\alpha_{i_k}, \alpha_{i_l})} z_l|, k < l)$ , admits symmetric meromorphic analytical continuation to  $(\mathbb{C}^*)^n$  with simple poles at shifted diagonals  $z_k = q^{\pm(\alpha_{i_k}, \alpha_{i_l})} z_l$ . Here symmetric group exchanges simultaneously the variables and their root labels. For instance, for  $n = 2$  it means that the function  $t_{\alpha}(z) t_{\beta}(w)$  has two simple poles at  $z = q^{(\alpha, \beta)} w$  if  $(\alpha, \beta) \neq 0$  and is commutative

$$t_{\alpha}(z) t_{\beta}(w) = t_{\beta}(w) t_{\alpha}(z)$$

in a sense of analytical continuation.

Let now  $\Pi$  be a set of simple positive roots of  $\mathfrak{g}$  and  $I$  be a finite set  $k_1, k_2, \dots, k_n$  of integers equipped with labels of simple roots, that is,  $I$  is a finite subset  $\check{I} = \{k_1, k_2, \dots, k_n\} \subset \mathbb{N}$  of a set of positive integers together with a map  $\iota: \check{I} \rightarrow \Pi$ . Let  $X_I$  be the following stratified space. As a total space,  $X_I$  is isomorphic to  $\mathbb{C}^n$  with coordinates  $z_k, k \in \check{I}$ . The closures of the strata are given by the intersections of hyperplanes  $H_{k,l} = \{z_k = q^{(\iota(k), \iota(l))} z_l\}$  for any  $k, l \in \check{I}$  such that  $(\iota(k), \iota(l)) \neq 0$  and  $H_i = \{z_i = 0, i \in \check{I}\}$ . By  $U_I$  we denote an open stratum: the complement to the union of hyperplanes.

Among all the strata there are the distinguished ones, which we call Serre strata. Let  $\alpha$  and  $\beta$  be two simple adjacent roots,  $a_{\alpha, \beta}$  be corresponding element of Cartan matrix,  $m_{\alpha, \beta} = 1 - a_{\alpha, \beta}$ . Let  $z_{l_1}, \dots, z_{l_m}$  be the coordinates of  $X_I$ , labeled by  $\alpha$ , that is  $\iota(l_i) = \alpha, i = 1, \dots, m$  and  $z_{l_0}$  be a coordinate labeled by  $\beta$ :  $\iota(l_0) = \beta$ . Then the stratum

$$z_{l_0} = q^{-\frac{m(\alpha, \alpha)}{2}} z_{l_1} = q^{-\frac{(m-2)(\alpha, \alpha)}{2}} z_{l_2} = \dots = q^{\frac{(m-2)(\alpha, \alpha)}{2}} z_{l_{m-1}} = q^{\frac{m(\alpha, \alpha)}{2}} z_{l_m} \quad (3.7)$$

is called Serre stratum. Another type of Serre strata appear due to the vanishing of the squares of the same fields in the same point. They have a form

$$z_{m_1} = z_{m_2} = q^{\pm(\iota(m_1), \iota(m_3))} z_{m_3} \quad (3.8)$$

for any  $m_1, m_2, m_3$  such that  $\iota(m_1) = \iota(m_2)$ .



We are interested in the integrals of the  $n$ -forms which have simple poles at the hyperplanes  $H_{k,l}$  and meromorphic singularities at hyperplanes  $H_k$  over certain  $n$ -cycles in  $U_I$ :

$$\omega = \frac{P(z, z^{-1})}{(2\pi i)^n \prod_{l \neq m} (z_{k_l} - q^{(\iota(k_l), \iota(k_m))} z_{k_m})} \frac{dz_{k_1}}{z_{k_1}} \wedge \frac{dz_{k_2}}{z_{k_2}} \wedge \dots \wedge \frac{dz_{k_n}}{z_{k_n}} \quad (3.9)$$

where  $P(z, z^{-1})$  is a polynomial over  $z_{k_i}$  and  $z_{k_i}^{-1}$ ,  $n = |\tilde{I}|$ .

The description of the homologies of the complement to the arrangement of hyperplanes is well known, see, e.g. [SV]. It looks as follows. Suppose we have a collection of hyperplanes  $H_i$  in a space  $X = \mathbb{C}^N$ . Hyperplanes define on  $X$  the structure of stratified space; the (closed) strata are all possible intersections of hyperplanes. A sequence

$$L_n \subset L_{n-1} \subset \dots \subset L_0$$

of strata is called a (full) flag of length  $n$ , if  $\text{codim}(L_k) = k$

Let  $U$  be a complement to the union of hyperplanes. The space  $H_n(U, \mathbb{C})$  is isomorphic to a factor of vectorspace, generated by all flags of length  $n$  modulo the Orlik-Solomon relations, attached to any incomplete flag  $L_n \subset \dots \subset L_{k+1} \subset L_{k-1} \dots \subset L_0$ ,  $\text{codim}(L_i) = i$ . The relation is

$$\sum_{\substack{L_k \subset L_{k+1} \subset L_k \subset L_{k-1} \\ \text{codim } L_k = k}} L_n \subset \dots \subset L_{k+1} \subset L_k \subset L_{k-1} \dots \subset L_0 = 0.$$

The cycle, corresponding to a flag  $L_n \subset L_{n-1} \subset \dots \subset L_0$  is  $n$ -dimensional torus, which is described as follows: first we take a small circle with a center in a generic point of  $L_n$ , surrounding  $L_n$  inside  $L_{n-1}$ , then for any point of this circle we draw a circle surrounding  $L_{n-1}$  inside  $L_{n-2}$  such that it does not intersect other strata inside  $L_{n-1}$  and so on. Note that in this procedure the next circle is much smaller than the previous one. The resulting torus lies in  $U$  and its orientation is given by the order in the procedure above.

For any cycle  $L = \{L_n \subset L_{n-1} \subset \dots \subset L_0\}$  and for any  $m$ -form  $\omega$ , regular in  $U$  we denote by  $\text{Res}_L \omega$  the integral of the form  $\omega$  over the cycles, defined by  $L$  (we take into account their dependence over generic point of  $L_n$ ). It is an  $(m-n)$ -form on the open part of  $L_n$ .

Return now to stratified space  $X_I$ . Denote by  $\Omega_I$  the space of such forms (3.9), for which any repeated residue to any Serre stratum (3.7), (3.8) is zero. Namely, we say that  $\omega \in \Omega_I$  if  $\text{Res}_L \omega = 0$  for any  $L = \{L_n \subset L_{n-1} \subset \dots \subset L_0\}$ , such that  $L_n$  is a Serre stratum (3.7), (3.8). We call such forms admissible.

The main goal of this subsection is the definition of factorizable system of antisymmetric cycles for any quantum affine algebra  $U_q(\mathfrak{g})$ . This definition means an assignment to any labeled set  $I$  a symmetric  $n$ -cycle  $D_I \in H_n(U_I)$ , where  $n = |I|$ , with certain factorization properties. Let us first explain symmetricity condition.

Any permutation  $\sigma \in S_n$  where  $n = |I|$  defines a new ordered set  $\sigma(I)$  with induced labeling. Moreover,  $\sigma$  defines a diffeomorphism of configuration spaces:  $\sigma : X_I \rightarrow X_{\sigma(I)}$ . We demand that:

$$\oint_{D_{\sigma(I)}} \omega = \oint_{D_I} \sigma^*(\omega) \quad (3.10)$$

for any  $\sigma \in S_{|I|}$  and  $\omega \in \Omega_{\sigma(I)}$ . Equivalently,

$$\oint_{D_{\sigma(I)}} \omega = (-1)^{l(\sigma)} \oint_{\sigma(D_I)} \omega,$$

which means that the cycle  $D_I$  is defined on the factor of  $U_I$  over the action of the product  $S_{i_1} \times S_{i_2} \times \dots \times S_{i_r}$  of symmetric groups, where  $i_k = \#k \in \tilde{I}$ ,  $\iota(k) = \alpha_k$  and can be labeled by an element of the lattice of positive roots of  $\mathfrak{g}$ .

Suppose now that finite labeled set  $I$ ,  $|I| = n + m$  is presented as a disjoint union of its labeled subsets  $I = I^1 \sqcup I^2$ ,  $|I^1| = n$ ,  $|I^2| = m$ . Then for any  $D_{I^1} \in H_n(U_{I^1}, \mathbb{C})$  and for any  $D_{I^2} \in H_m(U_{I^2}, \mathbb{C})$  we can attach the cycle  $D_{I^1} \times D_{I^2} \in H_{n+m}(U_I, \mathbb{C})$  as follows. There is a natural map  $\phi : X_{I^1} \times X_{I^2} \rightarrow X_I$ . The cycle  $D_{I^2}$

is  $m$ -dimensional compact manifold, and we can apply to it arbitrary dilatation  $z_l \rightarrow \varepsilon z_l, l \in \check{I}^2, \varepsilon > 0$  without changing homology class. We choose such small  $\varepsilon$  that all the points of the direct product of dilated  $D_{I^1}$  and of  $D_{I^2}$  belong to  $U_I$  under the image of  $\phi$ . It is possible by the following reasons: on every  $D_{I^j}$  absolute values  $|z_k|$  of the coordinates are restricted from both sides, which follows from the compactness of the cycles and from the observations that coordinate hyperplanes  $z_k = 0$  should not intersect the cycles by their definition. Then it is clear that for small enough  $\varepsilon$  a direct product of dilated  $D_{I^1}$  and of  $D_{I^2}$  does not intersect hyperplanes  $z_k = q^{\pm(\iota(k), \iota(l))} z_l$ , where  $k \in \check{I}^1$  and  $l \in \check{I}^2$ . But this is the only thing we are checking. This construction defines invariantly the cycle  $D_{I^1} \times D_{I^2} \in H_{n+m}(U_I, \mathbb{C})$  equipped with natural orientation.

Let us choose any form  $\omega \in \Omega_I$  satisfying additional property: it has no singularities at hyperplanes  $\{z_k = q^{\iota(k), \iota(l)} z_l\}$  for all  $k \in I^1$  and  $l \in I^2$ . Denote the space of such  $\omega$  by  $\Omega_{I^1, I^2}$ . We demand that the integral of any  $\omega \in \Omega_{I^1, I^2}$  over  $D_I$  coincides with the integral over  $D_{I^1} \times D_{I^2}$ :

$$\oint_{D_I} \omega = \oint_{D_{I^1} \times D_{I^2}} \omega \quad (3.11)$$

The factorization conditions mean that, first, the relation (3.11) holds for any decompositions  $I = I^1 \amalg I^2$  and for any  $\omega \in \Omega_{I^1, I^2}$  and, second, the initial conditions

$$D_{\{k\}} = \{|z_k| = 1\} \quad (3.12)$$

take place for any one point set  $I$ ,  $\check{I} = \{k\}$ .

Let  $q^n \neq 1$  for any natural  $n$ . We suggest the following

**Conjecture** For any  $U_q(\hat{\mathfrak{g}})$ , where  $\mathfrak{g}$  is simple Lie algebra, there exists a factorizable system  $D_I$  of antisymmetric cycles with initial conditions (3.12).

Moreover, we suppose that the cycles are unique as the functionals over admissible forms. Also, we suppose that they are uniquely defined by factorization conditions; the symmetricity should follow from the factorization properties.

We can describe precisely such a system for the algebra  $U_q^{(D)}(\hat{\mathfrak{sl}}_2)$ . In this case we have no Serre restrictions (3.7) and, since there is only one simple root for Lie algebra  $\mathfrak{g}$ , the cycles  $D_I$  are parameterized by positive integers  $n$ , so we denote them as  $D_n$ . They look as follows. Fix some order of the variables  $z_{k_1}, z_{k_2}, \dots, z_{k_n}$ . Then in the flag description the cycle  $D_n$  consists of all  $n$ -flags  $L = \{L_n \subset L_{n-1} \subset \dots \subset L_0\}$  in  $X_n = \mathbb{C}^n$ , all with multiplicity one, such that  $L_1$  is one of the hyperplanes

$$z_{k_1} = 0, \quad z_{k_1} = q^{-2} z_{k_j}, \quad j \neq 1,$$

$L_2$  is codimension two stratum, which can be obtained as an intersection of one of hyperplanes  $L_1$ , listed above, with hyperplanes

$$z_{k_2} = 0, \quad z_{k_2} = q^{-2} z_{k_j}, \quad j \neq 2$$

and so on. Clear, that in the last stage  $L_n$  coincides with an origin in  $\mathbb{C}^n$ .

**Proposition 3.2** The cycles  $D_n$  form factorizable system of antisymmetric cycles for  $U_q(\hat{\mathfrak{sl}}_2)$ , if  $q$  is not root of 1.

There are two short proofs of the proposition. Note first that it is sufficient to prove the symmetricity condition only. Then the factorization property follows from the definition of the cycles  $D_I$ .

For the first proof, it sufficient to check that for  $|q| > 1$ , the cycles  $D_n$  for  $I = \{1, \dots, n\}$  are homotopic to the product of unit circles, which are clearly antisymmetric and the factorization property is also clear. For another proof, one can note that the integral of  $\omega$  over  $D_n$  coincides with Grothendick residue [GH] of  $\omega$  with respect to the system of divisors

$$f_i(z) = z_i \prod_{j \neq i} (z_i - q^{-2} z_j) = 0, \quad i = 1, 2, \dots, n. \quad (3.13)$$

The Grothendique residue is correctly defined for a system of divisors  $f_i(z) = 0$  with common intersection being a point  $\{z_i = 0\}$  and can be written in this case as an integral  $\oint_{\Gamma} \omega$ , where  $\Gamma = \{z : |f_i(z)| = \varepsilon\}$ . The system (3.13) satisfy these conditions. By the definition of the residue, it is antisymmetric. The factorization property follows then from the form of  $f_i(z)$ .

Note that the vanishing conditions on 'diagonal' Serre strata (3.8) were not used in the proof.

### 3.4 The properties of the universal $R$ -matrix and factorization of the cycles

The universal  $R$ -matrix  $R$  for a quasitriangular Hopf algebra  $\mathcal{A}$  is characterized by the properties

$$\Delta^{op}(a) = R\Delta(a)R^{-1} \quad (3.14)$$

for any  $a \in \mathcal{A}$  and

$$(\Delta \otimes \text{id})R = R_{13}R_{23}, \quad (\text{id} \otimes \Delta)R = R_{13}R_{12}. \quad (3.15)$$

Let now  $\mathcal{A} = U_q^{(D)}(\widehat{\mathfrak{g}})$  where  $\mathfrak{g}$  has rank  $r$  and  $D_I$  be a factorizable system of antisymmetric cycles. Put

$$\mathcal{R} = \mathcal{K}\overline{\mathcal{R}} \quad (3.16)$$

where  $\mathcal{K}$  is given in (3.2) and

$$\begin{aligned} \overline{\mathcal{R}} &= \mathcal{P}\overrightarrow{\text{exp}}_{\{D_I\}} \oint \underline{dz} \sum_{i=1, \dots, r} t_{\alpha_i}(z) = \\ &1 + \sum_{n>0} \frac{1}{n!} \sum_{I: \check{I}=\{1,2,\dots,n\}} \oint_{D_I} t_{\iota(1)}(z_1) t_{\iota(2)}(z_2) \cdots t_{\iota(n)}(z_n) \underline{dz_1} \wedge \cdots \wedge \underline{dz_n} \end{aligned} \quad (3.17)$$

Here and below  $\underline{dz_k}$  means  $\frac{dz_k}{2\pi i z_k}$ .

**Theorem 1** *For any factorizable system  $\{D_I\}$  of antisymmetric cycles the tensor (3.16) satisfy the properties (3.14) and (3.15) of the universal  $R$ -matrix for topological Hopf algebra  $U_q^{(D)}(\widehat{\mathfrak{g}})$  with comultiplication  $\Delta^{(1)}$ .*

Note first that the form  $t_{\iota(1)}(z_1) t_{\iota(2)}(z_2) \cdots t_{\iota(n)}(z_n) \underline{dz_1} \wedge \cdots \wedge \underline{dz_n}$  satisfy vanishing conditions on the Serre strata. It can be deduced from the Serre relations in a form, presented in [E]. We omit corresponding calculations. By antisymmetry of the cycles we can rewrite then the series (3.17) in the form

$$\overline{\mathcal{R}} = \sum_{n_1, \dots, n_r \geq 0} \frac{1}{n_1! \cdots n_r!} \oint_{D_{n_1, \dots, n_r}} t_{\iota(1)}(z_1) t_{\iota(2)}(z_2) \cdots t_{\iota(n)}(z_n) \underline{dz_1} \wedge \cdots \wedge \underline{dz_n}$$

where  $D_{\{n_i\}}$  is the cycle, corresponding to labeled set  $I$ ,  $\check{I} = \{1, \dots, n_1 + \cdots + n_r\}$  and  $\iota(k) = \alpha_i$  if  $n_1 + \cdots + n_{i-1} < k \leq n_1 + \cdots + n_i$  for any order  $\alpha_1, \dots, \alpha_r$  of simple roots of  $\mathfrak{g}$ .

Let us prove an equality (3.14) for  $x = e_{\alpha_1}(z)$ . The basic property of the tensor  $\mathcal{K}$ , which we use here, is

$$\Delta^{(1), op}(x) = \mathcal{K}\Delta^{(2)}(x)\mathcal{K}^{-1} \quad (3.18)$$

for all  $x \in U_q^{(D)}(\widehat{\mathfrak{g}})$ . We have ( $\Delta$  means  $\Delta^{(1)}$ ):

$$\mathcal{R}\Delta e_{\alpha_1}(z)\mathcal{R}^{-1} = \mathcal{K}\overline{\mathcal{R}}(e_{\alpha_1}(z) \otimes 1)(\overline{\mathcal{R}})^{-1}\mathcal{K}^{-1} + \mathcal{K}\overline{\mathcal{R}}(\psi_{\alpha_1}^-(zq^{\frac{c_1}{2}}) \otimes e_{\alpha_1}(zq^{c_1}))(\overline{\mathcal{R}})^{-1}\mathcal{K}^{-1}.$$

Denote the summands of r.h.s. as  $X_1 = \mathcal{K}\overline{\mathcal{R}}(e_{\alpha_1}(z) \otimes 1)(\overline{\mathcal{R}})^{-1}\mathcal{K}^{-1}$  and  $X_2 = \mathcal{K}\overline{\mathcal{R}}(\psi_{\alpha_1}^-(zq^{\frac{c_1}{2}}) \otimes e_{\alpha_1}(zq^{c_1}))(\overline{\mathcal{R}})^{-1}\mathcal{K}^{-1}$ . We see from (2.5) and (2.8) that ( $n = n_1 + \dots + n_r$ )

$$X_1 - \mathcal{K}(e_{\alpha_1}(z) \otimes 1)\mathcal{K}^{-1} = \mathcal{K} \sum_{\substack{n_2, \dots, n_r \geq 0 \\ n_1 \geq k > 0}} \frac{1}{n_1! \cdots n_r!} \oint_{D_{\{n_i\}}} \underline{dz_1} \wedge \dots \wedge \underline{dz_n} t_{\alpha_1}(z_1) \cdots t_{\alpha_1}(z_{k-1})$$

$$\begin{aligned}
& \left( \left( \delta\left(\frac{z}{z_k} q^{-c}\right) \psi_{\alpha_1}^+(z_k q^{\frac{1}{2}c}) - \delta\left(\frac{z}{z_k} q^c\right) \psi_{\alpha_1}^-(z_k q^{\frac{1}{2}c}) \right) \otimes e_{\alpha_1}(z_i) \right) t_{\alpha_1}(z_{k+1}) \cdots t_{\alpha_1}(z_{n_1}) \prod_{\iota(j)>1} t_{\iota(j)}(z_j) \overline{\mathcal{R}}^{-1} \mathcal{K}^{-1} = \\
& \mathcal{K} \sum_{\substack{n_2, \dots, n_r \geq 0 \\ n_1 > 0}} \frac{n_1}{n_1! \cdots n_r!} \oint_{D_{\{n_i\}}} \delta\left(\frac{z}{z_1 q^c}\right) \psi_{\alpha_1}^+(z_1 q^{\frac{1}{2}c}) \otimes e_{\alpha_1}(z_1) t_{\alpha_1}(z_2) \cdots t_{\alpha_1}(z_{n_1}) \prod_{\iota(j)>1} t_{\iota(j)}(z_j) \underline{dz_1} \wedge \dots \wedge \underline{dz_n} \overline{\mathcal{R}}^{-1} \mathcal{K}^{-1} \\
& - \mathcal{K} \sum_{\substack{n_2, \dots, n_r \geq 0 \\ n_1 > 0}} \frac{n_1}{n_1! \cdots n_r!} \oint_{D_{\{n_i\}}} t_{\alpha_1}(z_2) \cdots t_{\alpha_1}(z_{n_1}) \prod_{\iota(j)>1} t_{\iota(j)}(z_j) \delta\left(\frac{z q^c}{z_1}\right) \psi_{\alpha_1}^-(z q^{\frac{1}{2}c}) \otimes e_{\alpha_1}(z_1) \underline{dz_1} \wedge \dots \wedge \underline{dz_n} \overline{\mathcal{R}}^{-1} \mathcal{K}^{-1}
\end{aligned}$$

due to antisymmetry of the cycle  $D_{\{n_i\}}$ . From the commutation relations (2.3), (2.4), (2.5) it follows that the integrands belong to a class  $\Omega_{I_{\{n_i\}}}$  and, moreover, the first integrand has no singularities at  $z_j = q^{(\alpha_1, \iota(j))} z_1$  for all  $j \neq 1$  and the second integrand has no singularities at  $z_1 = q^{(\alpha_1, \iota(j))} z_j$  for all  $j \neq 1$ . From factorization condition and from the definition of the multiplication of the currents together with initial condition (3.12) we see that

$$\begin{aligned}
& X_1 - \mathcal{K}(e_{\alpha_1}(z) \otimes 1) \mathcal{K}^{-1} = \\
& \mathcal{K} \oint \underline{dz_1} \delta\left(\frac{z}{z_1 q^c}\right) \psi_{\alpha_1}^+(z_1 q^{\frac{1}{2}c}) \otimes e_{\alpha_1}(z_1) \cdot \sum_{\substack{n_2, \dots, n_r \geq 0 \\ n_1 > 0}} \frac{1}{(n_1 - 1)! \cdots n_r!} \oint_{D_{n_1-1, n_2, \dots, n_r}} \prod_{i>1} t_{\iota(i)}(z_i) \underline{dz_2} \wedge \dots \wedge \underline{dz_n} \overline{\mathcal{R}}^{-1} \mathcal{K}^{-1} \\
& - \mathcal{K} \sum_{\substack{n_2, \dots, n_r \geq 0 \\ n_1 > 0}} \frac{1}{(n_1 - 1)! \cdots n_r!} \oint_{D_{n_1-1, n_2, \dots, n_r}} \prod_{i>1} t_{\iota(i)}(z_i) \underline{dz_2} \wedge \dots \wedge \underline{dz_n} \cdot \oint \underline{dz_1} \delta\left(\frac{z q^c}{z_1}\right) \psi_{\alpha_1}^-(z q^{\frac{1}{2}c}) \otimes e_{\alpha_1}(z_1) \overline{\mathcal{R}}^{-1} \mathcal{K}^{-1} \\
& = \mathcal{K} \psi_{\alpha_1}^+(z q^{-\frac{c_1}{2}}) \otimes e_{\alpha_1}(z q^{-c_1}) \mathcal{K}^{-1} - \mathcal{K} \overline{\mathcal{R}} \psi_{\alpha_1}^-(z q^{\frac{c_1}{2}}) \otimes e_{\alpha_1}(z q^{c_1}) \overline{\mathcal{R}}^{-1} \mathcal{K}^{-1}
\end{aligned}$$

after a change of the index of summation over  $n_1$ . The last summand cancels  $X_2$  so

$$\mathcal{R} \Delta e_{\alpha_1}(z) \mathcal{R} = \mathcal{K}(e_{\alpha_1}(z) \otimes 1) \mathcal{K}^{-1} + \mathcal{K} \psi_{\alpha_1}^+(z q^{-\frac{c_1}{2}}) \otimes e_{\alpha_1}(z q^{-c_1}) \mathcal{K}^{-1} = \mathcal{K} \Delta^{(2)}(e_{\alpha_1}(z)) \mathcal{K}^{-1} = \Delta^{(1), op}(e_{\alpha_1}(z))$$

which finishes the proof.

Let us prove one of the relations (3.15). The left hand side equals to

$$(\Delta \otimes 1) \mathcal{K} \overline{\mathcal{R}} = \mathcal{K}_{13} \mathcal{K}_{23} (\Delta \otimes 1) \overline{\mathcal{R}} = \mathcal{K}_{13} \mathcal{K}_{23} \mathcal{P} \overrightarrow{\text{exp}}_{\{D_I\}} \oint \underline{dz} \sum_{i=1, \dots, r} (t_{\alpha_i}^1(z) + t_{\alpha_i}^2(z)),$$

where

$$t_{\alpha_i}^1(z) = (q^{-1} - q) e_{-\alpha_i}(z q^{c_2}) \otimes \psi_{\alpha_i}^+(z q^{c_2/2}) \otimes e_{\alpha_i}(z) \quad (3.19)$$

and

$$t_{\alpha_i}^2(z) = (q^{-1} - q) \cdot 1 \otimes e_{-\alpha_i}(z) \otimes e_{\alpha_i}(z) \quad (3.20)$$

On the other side,

$$\begin{aligned}
& \mathcal{R}_{13} \mathcal{R}_{23} = \mathcal{K}_{13} \overline{\mathcal{R}}_{13} \mathcal{K}_{23} \overline{\mathcal{R}}_{23} = \mathcal{K}_{13} \mathcal{K}_{23} (\mathcal{K}_{23}^{-1} \overline{\mathcal{R}}_{13} \mathcal{K}_{23}) \overline{\mathcal{R}}_{23} = \\
& \mathcal{K}_{13} \mathcal{K}_{23} \mathcal{P} \overrightarrow{\text{exp}}_{\{D_I\}} \oint \underline{dz} \sum_{i=1, \dots, r} t_{\alpha_i}^1(z) \cdot \mathcal{P} \overrightarrow{\text{exp}}_{\{D_I\}} \oint \underline{dz} \sum_{i=1, \dots, r} t_{\alpha_i}^2(z)
\end{aligned}$$

due to the properties of  $\mathcal{K}$ . So we have to prove an equality

$$\mathcal{P} \overrightarrow{\text{exp}}_{\{D_I\}} \oint \underline{dz} \sum_{i=1, \dots, r} (t_{\alpha_i}^1(z) + t_{\alpha_i}^2(z)) = \mathcal{P} \overrightarrow{\text{exp}}_{\{D_I\}} \oint \underline{dz} \sum_{i=1, \dots, r} t_{\alpha_i}^1(z) \cdot \mathcal{P} \overrightarrow{\text{exp}}_{\{D_I\}} \oint \underline{dz} \sum_{i=1, \dots, r} t_{\alpha_i}^2(z) \quad (3.21)$$

where  $t_{\alpha_i}^1(z)$  and  $t_{\alpha_i}^2(z)$  are given by (3.19) and (3.20). An equality (3.21) is equivalent to following equalities of integrals. Let  $I$ ,  $\tilde{I} = \{1, \dots, n\}$  be a labeled set and  $j(1), \dots, j(n)$  be arbitrary sequence of numbers 1 and 2,  $j(k) = 1, 2$ . Then we should have:

$$\oint_{D_I} t_{\iota(1)}^{j(1)}(z_1) t_{\iota(2)}^{j(2)}(z_2) \cdots t_{\iota(n)}^{j(n)}(z_n) \underline{dz_1} \wedge \cdots \wedge \underline{dz_n} =$$

$$\oint_{D_I^1} t_{\iota(k_1)}^1(z_{k_1}) \cdots t_{\iota(k_m)}^1(z_{k_m}) \underline{dz_{k_1}} \wedge \cdots \wedge \underline{dz_{k_m}} \cdot \oint_{D_I^2} t_{\iota(k_{m+1})}^2(z_{k_{m+1}}) \cdots t_{\iota(k_n)}^2(z_{k_n}) \underline{dz_{k_{m+1}}} \wedge \cdots \wedge \underline{dz_{k_n}} \quad (3.22)$$

where  $I^1 = \{k_1, \dots, k_m; j(k_i) = 1\}$  and  $I^2 = \{k_{m+1}, \dots, k_n; j(k_i) = 2\}$ . Again, we see from commutation relations (2.6), that the form  $\omega = t_{\iota(1)}^{j(1)}(z_1) t_{\iota(2)}^{j(2)}(z_2) \cdots t_{\iota(n)}^{j(n)}(z_n) \underline{dz_1} \wedge \cdots \wedge \underline{dz_n}$  belongs to a class  $\Omega_I$ , and, moreover, has no singularities at hyperplanes  $\{z_k = q^{\iota(k), \iota(l)} z_l\}$  for all  $k \in I^2$  and  $l \in I^1$  and the equality (3.22) holds due to factorization conditions on the cycles.

The two natural questions immediately appear for the  $R$ -matrix (3.16). First, to which representations it can be applied, and second, how it is connected to the  $R$ -matrix (3.3), constructed by means of Cartan-Weyl generators. Let us first try to apply (3.16) to a tensor product  $W \otimes V$  of the graded representations with highest degree. Then there is a correct action of  $\mathcal{K}$  and of each term in the series (3.17) (for instance, because the integral can be calculated by taking multiple residues which belong to completed algebra  $U_q^{(D)}(\widehat{\mathfrak{g}})$ ). We can conclude now that this action is well defined in a formal sense, that is, if we consider  $U_q^{(D)}(\widehat{\mathfrak{g}})$  as an algebra over  $\mathbb{C}[[q-1]]$  as well as the representations.

Let now  $W$  and  $V$  be highest weight representations. In particular, it means that they are graded modules of highest degree. In this case we know from Proposition 3.1 that the  $R$ -matrix (3.3) is well defined operator in  $W \otimes V$ . In particular, it means that  $\mathcal{R}^{CW}$  is also well defined operator in a formal sense, that is, over the ring  $\mathbb{C}[[q-1]]$ . By uniqueness argument we conclude that in a formal sense the actions of  $\mathcal{R}^{CW}$  and of  $\mathcal{R}$  coincide. So they coincide as formal power series over  $(q-1)$ , and the series (3.17) converge and its action coincide with the action of  $\mathcal{R}^{CW}$ . We summarize it in the following proposition.

**Proposition 3.3** *The action of  $R$ -matrix  $\mathcal{R}$  (3.16) on tensor product of highest weight representations is well defined and coincides with the action of  $\mathcal{R}^{CW}$  (3.3).*

## 4 Calculations for $U_q^{(D)}(\widehat{\mathfrak{sl}}_2)$

In this section we drop everywhere an index of a simple root of  $\mathfrak{sl}_2$  and denote  $e(z) \equiv e_\alpha(z)$  and  $f(z) \equiv e_{-\alpha}(z)$ . Following [DK] and general receipt of vertex operator algebras, we study first the fields, appearing as the poles in the products of generating functions. Surprisingly, we will see in the first subsection, that the algebra, generated by the descenders of the tensor field  $f(z) \otimes e(z)$ , has simple closed form (4.13). Next, we derive from the contour integral description the recurrence relations on the terms of the universal  $R$ -matrix for  $U_q^{(D)}(\widehat{\mathfrak{sl}}_2)$ , which can be written as simple differential equation. Its solution has a structure of certain vertex operator due to the simple structure of tensor fields. Finally, we present some application of the derived expression for the  $R$ -matrix.

### 4.1 Algebra of tensor fields

Let us first look to the relation (2.3), which now reads as  $(z_1 - q^2 z_2) e(z_1) e(z_2) = (q^2 z_1 - z_2) e(z_2) e(z_1)$ . It shows that the function  $e(z_1) e(z_2)$  has the only pole at the point  $z_1 = q^2 z_2$ . The residue could be calculated in two ways. Since the product  $e(z_1) e(z_2)$  is well defined in completed algebra  $U_q^{(D)}(\widehat{\mathfrak{sl}}_2)$ , the contour of integration can be replaced by two circles and, how it is explained in [DK], it gives the residue as a difference of formal integrals:

$$\text{res}_{z_1=q^2 z_2} e(z_1) e(z_2) \frac{dz_1}{z_1} = \oint e(z_1) e(z_2) \underline{dz_1} - \oint \frac{q^{-2} - z_1/z_2}{1 - q^{-2} z_1/z_2} e(z_2) e(z_1) \underline{dz_1}$$

which gives

$$\begin{aligned} \text{res}_{z=q^2 w} e(z) e(w) \frac{dz}{z} &= \sum_{n \in \mathbb{Z}} w^{-2n} q^{-2n} \left\{ (1 - q^{-2}) e_n^2 + (q^2 - q^{-2}) \sum_{k=1}^{\infty} q^{-2k} e_{n-k} e_{n+k} \right\} + \\ &+ (q^2 - q^{-2}) \sum_{n \in \mathbb{Z}} w^{-2n-1} q^{-2n-2} \sum_{k=0}^{\infty} q^{-2k} e_{n-k} e_{n+k+1} \end{aligned} \quad (4.1)$$

which is well defined series in completed algebra  $U_q^{(D)}(\widehat{\mathfrak{sl}}_2)$ . We can use also the rule for the calculation of poles of the first order:

$$\operatorname{res}_{z=q^2w} e(z)e(w)\frac{dz}{z} = \lim_{z \rightarrow q^2w} (1 - q^2w/z) e(z)e(w) = (q^2 - q^{-2})e(w)e(q^2w) \quad (4.2)$$

Again, we can see from commutation relations that the product  $e(z)e(w)e(q^2w)$  has unique pole, which equals, up to a constant, to  $e(w)e(q^2w)e(q^4w)$ . Let us define the following generating functions from  $U_q^{(D)}(\widehat{\mathfrak{sl}}_2)$   $e^{(n)}(z)$  and  $f^{(n)}(z)$  by induction as

$$e^{(n)}(z) = \operatorname{res}_{z_1=q^{2(n-1)}z} e(z_1)e^{(n-1)}(z)\frac{dz_1}{z_1}, \quad f^{(n)}(z) = \operatorname{res}_{z_1=q^{2(n-1)}z} f^{(n-1)}(z)f(z_1)\frac{dz_1}{z_1},$$

Put also

$$t^{(n)}(z) = - \operatorname{res}_{z_1=q^{2(n-1)}z} t(z_1)t^{(n-1)}(z)\frac{dz_1}{z_1}, \quad (4.3)$$

where, as before,  $t(z) = (q^{-1} - q)f(z) \otimes e(z)$ . and  $e^{(1)}(z)$  stands for  $e(z)$ ,  $f^{(1)}(z)$  for  $f(z)$ ,  $t^{(1)}(z)$  for  $t(z)$ . Then, analogously to (4.2),

$$e^{(n)}(z) = (q^{-1} - q)^{n-1} [n-1]_q! [n]_q! \tilde{e}^{(n)}(z), \quad f^{(n)}(z) = (q^{-1} - q)^{n-1} [n-1]_q! [n]_q! \tilde{f}^{(n)}(z), \quad (4.4)$$

$$t^{(n)}(z) = (q^{-1} - q)^{2n-1} [n-1]_q! [n]_q! \tilde{f}^{(n)}(z) \otimes \tilde{e}^{(n)}(z), \quad (4.5)$$

where

$$\tilde{e}^{(n)}(z) = e(z)e(q^2z) \cdots e(q^{2(n-1)}z), \quad \tilde{f}^{(n)}(z) = f(q^{2(n-1)}z) \cdots f(q^2z)f(z).$$

Iterating the calculations (4.1), one can describe these fields in a component form:

$$\tilde{e}^{(n)}(z) = \sum_{m \in \mathbb{Z}} (zq^{2n})^m \sum_{\substack{\lambda_1 \geq \dots \geq \lambda_n, \\ \lambda_1 + \dots + \lambda_n = m}} \frac{q^{-2(\lambda_1 + \dots + k\lambda_k + \dots + n\lambda_n)}}{\prod_{j \in \mathbb{Z}} (\lambda'_j - \lambda'_{j+1})_{q^2}!} e_{\lambda_n} e_{\lambda_{n-1}} \cdots e_{\lambda_1} \quad (4.6)$$

$$\tilde{f}^{(n)}(z) = \sum_{m \in \mathbb{Z}} (zq^{-2})^m \sum_{\substack{\lambda_1 \geq \dots \geq \lambda_n, \\ \lambda_1 + \dots + \lambda_n = m}} \frac{q^{-2(\lambda_1 + \dots + k\lambda_k + \dots + n\lambda_n)}}{\prod_{j \in \mathbb{Z}} (\lambda'_j - \lambda'_{j+1})_{q^{-2}}!} f_{\lambda_n} f_{\lambda_{n-1}} \cdots f_{\lambda_1} \quad (4.7)$$

Here  $\lambda'_j = \#k$ , such that  $\lambda_k \geq j$ ,  $j \in \mathbb{Z}$ . The product in denominator is finite, since there are only finitely many distinct  $\lambda'_j$  for a given choice of  $\lambda_k$ .

**Lemma 4.1** *The product  $e^{(n)}(z_1)e^{(m)}(z_2)$  has poles at the points*

$$z = \frac{z_1}{z_2} = \begin{cases} q^2, q^4, \dots, q^{2m}, & n \geq m \\ q^{2m-2n+2}, q^{2m-2n+4}, \dots, q^{2m}, & n < m, \end{cases} \quad (4.8a)$$

and zeroes at the points

$$z = \begin{cases} q^{-2n+2}, q^{-2n+4}, \dots, q^{-2n+2m}, & n \geq m \\ q^{-2n+2}, q^{-2n+4}, \dots, 1, & n < m. \end{cases} \quad (4.8b)$$

Analogously, the product  $f^{(n)}(z_1)f^{(m)}(z_2)$  has zeroes at the points

$$z = \begin{cases} 1, q^2, \dots, q^{2m-2}, & n \geq m \\ q^{2m-2n}, q^{2m-2n+2}, \dots, q^{2m-2}, & n < m, \end{cases} \quad (4.8c)$$

and simple poles at

$$z = \begin{cases} q^{-2n}, q^{-2n+2}, \dots, q^{-2n+2m-2}, & n \geq m \\ q^{-2n}, q^{-2n+2}, \dots, q^{-2}, & n < m. \end{cases} \quad (4.8d)$$

*Proof.* Let us prove the part of the lemma concerning the composed current  $e^{(n)}(z)$ . The rest can be proved analogously. From the fact that the product  $e(z_1)e(z_2)$  has simple pole at the point  $z_1 = q^2 z_2$  and simple zero at  $z_1 = z_2$  it is clear that the product  $e(q^{2k} z_1)e^{(m)}(z_2)$  has one simple pole at  $z_1 = q^{2(m-k)} z_2$  and one simple zero at  $z_1 = q^{-2k} z_2$ , for  $k = 0, 1, \dots, n-1$ . The rest poles and zeros cancel each others. When  $n < m$  this is the structure of poles and zeros given in (4.8a) and (4.8b). When  $n > m$  there is an additional poles/zeros cancellation. Namely, the poles at the points  $z_1 = q^{-2k} z_2$ ,  $k = 0, 1, \dots, n-m-1$  cancel with the zeros at the same points.

Put  $g'(z) = \frac{q^{-2}-z}{1-q^{-2}z}$ . Then we have for all  $k$ , such that  $\max(1, m-n+1) \leq k \leq m$ :

$$\begin{aligned} \operatorname{res}_{z_1=q^{2k}z_2} \tilde{e}^{(n)}(z_1)\tilde{e}^{(m)}(z_2)\frac{dz_1}{z_1} &= \lim_{z_1 \rightarrow q^{2k}z_2} (1 - q^{2k}z_2/z_1) \prod_{i=0}^{k-1} \prod_{j=0}^{n-1} g'(q^{2(j-i)}z_1/z_2) \tilde{e}^{(k)}(z_2) \tilde{e}^{(n)}(z_1) \tilde{e}^{(m-k)}(q^{2k}z_2) = \\ &= \lim_{z_1 \rightarrow q^{2k}z_2} (q^2 - q^{2(k-1)}z_2/z_1) \prod_{i=0}^{k-2} \prod_{j=0}^{n-1} g'(q^{2j-2i}z) \prod_{j=1}^{n-1} g'(q^{2j-2k+2}z) \tilde{e}^{(k)}(z_2) \tilde{e}^{(n)}(z_1) \tilde{e}^{(m-k)}(q^{2k}z_2) = \\ &= (q - q^{-1}) \prod_{j=0}^{k-1} \frac{[n+j]_q [n+j+1]_q}{[j]_q [j+1]_q} \tilde{e}^{(n+k)}(z_2) \tilde{e}^{(m-k)}(q^{2k}z_2), \end{aligned} \quad (4.9)$$

since  $g'(q^{2k}) = \frac{[k+1]_q}{[k-1]_q}$  for  $k = 2, 3, \dots$ .

Analogously, for all  $k$ , such that  $\max(1, n-m+1) \leq k \leq n$

$$\operatorname{res}_{z_1=q^{-2k}z_2} \tilde{f}^{(n)}(z_1)\tilde{f}^{(m)}(z_2)\frac{dz_1}{z_1} = (q^{-1} - q) \prod_{j=0}^{k-1} \frac{[m+j]_q [m+j+1]_q}{[j]_q [j+1]_q} \tilde{f}^{(n-k)}(z_2) \tilde{f}^{(m+k)}(q^{-2k}z_2). \quad (4.10)$$

These calculations together with (4.4) give the following commutation relations:

$$\begin{aligned} e^{(n)}(z_1)e^{(m)}(z_2) &= \prod_{k=0}^{m-1} \prod_{l=0}^{n-1} g'(q^{2(k-l)}z) e^{(m)}(z_2) e^{(n)}(z_1) + \\ &+ \sum_{k=1}^m [m]_q \begin{bmatrix} m-1 \\ k \end{bmatrix}_q \begin{bmatrix} m-1 \\ k-1 \end{bmatrix}_q \delta\left(\frac{z_1}{q^{2k}z_2}\right) e^{(n+k)}(z_2) e^{(m-k)}(z_1) \end{aligned} \quad (4.11)$$

Analogously, we can derive the relations

$$\begin{aligned} f^{(n)}(z_1)f^{(m)}(z_2) &= \prod_{k=0}^{m-1} \prod_{l=0}^{n-1} g(q^{2(k-l)}z) f^{(m)}(z_2) f^{(n)}(z_1) - \\ &- \sum_{k=1}^n [n]_q \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \delta\left(\frac{q^{2k}z_1}{z_2}\right) f^{(n-k)}(z_2) f^{(m+k)}(z_1) \end{aligned} \quad (4.12)$$

where  $g(z) = \frac{q^2-z}{1-q^2z}$ . Note first, that the coefficients before delta functions are the new fields, and second, that some of these fields in r.h.s. of (4.11), (4.12) equal zero according to the structure of zeroes (4.8).

Lemma 4.1 shows also that the only (simple) poles of the product  $t^{(n)}(z_1)t^{(m)}(z_2)$  are  $z_1 = q^{2m}z_2$  and  $z_1 = q^{-2n}z_2$  and from (4.9) and (4.10) we deduce that

$$[t^{(n)}(z_1), t^{(m)}(z_2)] = \delta(q^{2n}z_1/z_2) t^{(n+m)}(z_1) - \delta(q^{-2m}z_1/z_2) t^{(n+m)}(z_2). \quad (4.13)$$

so the tensor fields  $t^{(n)}(z)$  form a closed algebra. In particular, from (4.13) we have the following

**Proposition 4.1** *The total integrals of the fields  $t^{(n)}(z)$  around infinity*

$$I^{(n)} = \oint t^{(n)}(z) dz$$

commute between themselves:

$$[I^{(n)}, I^{(m)}] = 0. \quad (4.14)$$

## 4.2 Calculation of the contour integrals

We want to calculate the integral

$$\overline{\mathcal{R}}^{(n)} = \frac{1}{n!} \oint_{D_n} t^{(1)}(z_1) \underline{dz_1} \cdots t^{(1)}(z_n) \underline{dz_n}$$

over the cycles described in the previous section. It was noted in the proof of Proposition (3.2), that for  $|q| > 1$  the cycle  $D_n$  is homotopic to a product of unit circles.

We can also deform it slightly to a torus  $C_i = \{|z_i| = r_i\}$  in such a way, that it has no intersections with diagonals. Let us consider it as the multiple integral

$$\overline{\mathcal{R}}^{(n)} = \frac{1}{n!} \oint_{C_n} \cdots \oint_{C_1} t^{(1)}(z_1) \underline{dz_1} \cdots t^{(1)}(z_n) \underline{dz_n}$$

and integrate first over  $z_1$  for fixed other  $z_j$ . We can move the contour  $C_1$  to infinity crossing the poles  $z_1 = q^2 z_j$ ,  $j = 2, \dots, n$ . The residue at the pole  $z_1 = q^2 z_2$  is equal, due to (4.9), to <sup>4</sup>

$$- \oint_{C_2} t^{(2)}(z_2) \underline{dz_2} t^{(1)}(z_3) \underline{dz_3} \cdots t^{(1)}(z_n) \underline{dz_n}. \quad (4.15)$$

For the calculation of the residues at  $z_1 = z_3$  we first use the commutativity of analytical continuations of  $t^{(k_i)}(z_i)$ , which mean, in particular, that they commute on the contour which does not cross their singularities and then repeat the same calculation as for  $z_1 = q^2 z_2$ . As a result, we will have  $(n-1)$  integrals with different contours, obtained by permutations of each other, but due to the absence of the poles near diagonal (see (4.13)), all they are equivalent.

So, we have,

$$\begin{aligned} \overline{\mathcal{R}}^{(n)} = \frac{1}{n!} & \left( \oint_{C_1} t^{(1)}(z_1) \underline{dz_1} \cdot \oint_{C_n} \cdots \oint_{C_2} t^{(1)}(z_2) \underline{dz_2} \cdots t^{(1)}(z_n) \underline{dz_n} + \right. \\ & \left. + (n-1) \oint_{C_{n-1}} \cdots \oint_{C_2} t^{(2)}(z_2) \underline{dz_2} t^{(1)}(z_3) \underline{dz_3} \cdots t^{(1)}(z_n) \underline{dz_n} \right) \end{aligned}$$

or

$$\overline{\mathcal{R}}^{(n)} = \frac{1}{n} I^{(1)} \overline{\mathcal{R}}^{(n-1)} + \frac{n-1}{n!} \oint_{C_n} \cdots \oint_{C_2} t^{(2)}(z_2) \underline{dz_2} t^{(1)}(z_3) \underline{dz_3} \cdots t^{(1)}(z_n) \underline{dz_n}.$$

Again, integrate the second integral over  $z_2$  for fixed others. We see, that this integral has singularities for  $z_2 = q^{\pm 4} z_j$ . It is clear, that when we move the contour to infinity, we cross the poles  $z_2 = q^4 z_j$ ,  $j = 3, \dots, n$ . By the same trick we have now

$$\begin{aligned} \overline{\mathcal{R}}^{(n)} = \frac{1}{n} I^{(1)} \overline{\mathcal{R}}^{(n-1)} + \frac{n-1}{n!} & \oint_{C_2} t^{(2)}(z_2) \underline{dz_2} \cdot \oint_{C_{n-1}} \cdots \oint_{C_3} t^{(1)}(z_3) \underline{dz_3} \cdots t^{(1)}(z_n) \underline{dz_n} + \\ & + \frac{(n-1)(n-2)}{n!} \oint_{C_n} \cdots \oint_{C_3} t^{(3)}(z_3) \underline{dz_3} t^{(1)}(z_4) \underline{dz_4} \cdots t^{(1)}(z_n) \underline{dz_n} = \end{aligned}$$

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<sup>4</sup>Expanding the contour to infinity we should pick up minus residue. The minus sign in the formula (4.15) results that in all calculations below we avoid the appearing of the alternating signs.



$$= \frac{1}{n} \left( I^{(1)} \overline{\mathcal{R}}^{(n-1)} + I^{(2)} \overline{\mathcal{R}}^{(n-2)} + \frac{(n-1)(n-2)}{n!} \oint_{C_n} \dots \oint_{C_3} t^{(3)}(z_3) \underline{dz}_3 t^{(1)}(z_4) \underline{dz}_4 \dots t^{(1)}(z_n) \underline{dz}_n \right).$$

The inductive calculation give the following recurrence relation:

$$\overline{\mathcal{R}}^{(n)} = \frac{1}{n} \left( I^{(1)} \overline{\mathcal{R}}^{(n-1)} + I^{(2)} \overline{\mathcal{R}}^{(n-2)} + \dots + I^{(n)} \overline{\mathcal{R}}^{(0)} \right) \quad (4.16)$$

with initial condition  $\overline{\mathcal{R}}^{(0)} = 1 \otimes 1$ . For the solution of this recurrence relation let us introduce generating functions

$$\overline{\mathcal{R}}(u) = \sum_{n \geq 0} \overline{\mathcal{R}}^{(n)} u^n, \quad I(u) = \sum_{n \geq 1} I^{(n)} u^n.$$

Then the relation (4.16) means the following differential equation:

$$u \frac{d}{du} \overline{\mathcal{R}}(u) = I(u) \overline{\mathcal{R}}(u) \quad (4.17)$$

with initial condition  $\overline{\mathcal{R}}(0) = 1 \otimes 1$ . Its solution for the commutative variable  $I(u)$  is

$$\overline{\mathcal{R}}(u) = \exp \left( \int_0^u \frac{I(v)}{v} dv \right),$$

This proves the following

**Theorem 2** For the algebra  $U_q^{(D)}(\widehat{\mathfrak{sl}}_2)$  the factor  $\overline{\mathcal{R}}$  of the universal  $R$ -matrix (3.16) can be written in the form

$$\overline{\mathcal{R}} = \overline{\mathcal{R}}(1) = \exp \left( \sum_{n \geq 1} \frac{I^{(n)}}{n} \right) = \exp \left( \sum_{n \geq 1} \oint \frac{dz}{2\pi i z} \frac{t^{(n)}(z)}{n} \right). \quad (4.18)$$

### 4.3 Some applications

Let us first apply the formula (4.18) to the integrable representations of  $U_q^{(D)}(\widehat{\mathfrak{sl}}_2)$ . It is known [DM], that in integrable representation of  $U_q^{(D)}(\widehat{\mathfrak{sl}}_2)$  of level  $k$  all the currents  $e^{(n)}(z)$  and  $f^{(n)}(z)$  vanish for  $n > k$ . It gives immediately the following statement:

**Proposition 4.2** The action of the universal  $R$ -matrix for  $U_q^{(D)}(\widehat{\mathfrak{sl}}_2)$  in tensor product  $V \otimes W$  of highest weight representations, where one of them is integrable of level  $k$ , coincides with the action of

$$\mathcal{K} \exp \left( \sum_{n=1}^k \oint \frac{dz}{2\pi i z} \frac{t^{(n)}(z)}{n} \right).$$

In particular, for  $k = 1$ , it is

$$\mathcal{K} \exp \left( (q^{-1} - q) \oint \frac{dz}{2\pi i z} f(z) \otimes e(z) \right).$$

Here  $\mathcal{K}$  is given by (3.4).

We can deduce from (4.18) the formula for the universal  $R$ -matrix of  $U_q(\mathfrak{sl}_2)$ , which is well known [D1]. To do this, we suppose that we act by  $\mathcal{R}$  on a tensor product of highest weight representations of zero level with zero top degree and calculate the matrix coefficient between tensor products of the vectors of zero degree. Then in the expression of the  $R$ -matrix only the terms composed from the elements from  $U_q(\mathfrak{sl}_2)$   $e_0$ ,  $f_0$ ,  $q^h$  and  $q^{-h}$

survive. What is left, is to count the coefficients before  $e_0^n$  and  $f_0^n$  in the fields  $e^{(n)}$  and  $f^{(n)}$ . They are given in (4.6), (4.7) and (4.4). Finally, we get

$$R = q^{\frac{-h \otimes h}{2}} \exp \left( \sum_{n \geq 1} \frac{(q^{-1} - q)^{2n-1}}{n[n]_q} f^n \otimes e^n \right).$$

One can check that it coincides with the usual presentation of the universal  $R$ -matrix via  $q^2$ -exponential function using, for instance, presentation of  $q$ -exponent in a form of infinite product.

Finally, we can use factorized expression of [DK] for the universal  $R$ -matrix and substitute there (4.18) in every factor. As a result, we get an expression of the form, which depends on a fixed normal ordering  $>$  of the system  $\Delta_+$  of positive roots (or, equivalently, on the reduced decomposition of the longest element of the Weyl group) of simple Lie algebra  $\mathfrak{g}$  of simple laced type:

$$\mathcal{R} = \mathcal{K} \prod_{\gamma \in \Delta_+}^{\rightarrow} \exp \left( \sum_{n \geq 1} \oint \frac{dz}{2\pi i z} \frac{t_\gamma^{(n)}(z)}{n} \right)$$

where  $\Delta_+$  is the system of positive roots of simple Lie algebra  $\mathfrak{g}$ , the currents  $e_\gamma(z)$  and  $e_{-\gamma}(z)$  are built as in [DK],  $t_\gamma(z) = (q^{-1} - q)e_{-\gamma}(z) \otimes e_\gamma(z)$  and the currents  $t_\gamma^{(n)}(z)$  are constructed from  $e_\gamma(z)$  and  $e_{-\gamma}(z)$  according to the rules (4.4). Again, we can restrict ourselves to zero level matrix coefficients and recover precisely multiplicative formula for the  $R$ -matrix of  $U_q(\mathfrak{g})$ , see, e.g., [KT1].

We can also repeat our considerations for Yangians and elliptic algebras, as it was done in [DK] and in [DKKP]. Another applications are given in [DKP]

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